



TITLE:

Discrete series for semisimple symmetric spaces(WORKSHOP ON ALGEBRAIC GROUPS AND RELATED TOPICS)

AUTHOR(S):

MATSUKI, Toshihiko

CITATION:

MATSUKI, Toshihiko. Discrete series for semisimple symmetric spaces(WORKSHOP ON ALGEBRAIC GROUPS AND RELATED TOPICS). 数理解析研究所講究録 1990, 737: 1-5

ISSUE DATE:

1990-12

URL:

<http://hdl.handle.net/2433/102051>

RIGHT:

Discrete series for semisimple symmetric spaces

Toshihiko MATSUKI

The aim of this note is to explain the essence of the theory of discrete series for semisimple symmetric spaces $X = G/H$ in [7], [6] and [3].

Let \mathfrak{g} be a real semisimple Lie algebra and \mathfrak{g}_c the complexification of \mathfrak{g} . Let σ be an involution ($\sigma^2 = id.$) of \mathfrak{g} and θ a Cartan involution of \mathfrak{g} such that $\sigma\theta = \theta\sigma$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be $+1, -1$ -eigenspace decompositions for σ and θ , respectively. Then

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{s} \cap \mathfrak{h}) \oplus (\mathfrak{s} \cap \mathfrak{q}).$$

Put $\mathfrak{h}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{k} \cap \mathfrak{q})$, $\mathfrak{k}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{s} \cap \mathfrak{h})$ and $\mathfrak{g}^d = \mathfrak{h}^d + \mathfrak{k}^d + (\mathfrak{s} \cap \mathfrak{q})$. Then $(\mathfrak{h}^d)_c = \mathfrak{k}_c$, $(\mathfrak{k}^d)_c = \mathfrak{h}_c$ and $(\mathfrak{g}^d)_c = \mathfrak{g}_c$.

Let G_c be a connected Lie group with Lie algebra \mathfrak{g}_c . Let $G, K, H, G^d, K^d, H^d, K_c$ and H_c be the analytic subgroups of G_c for $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{h}^d, \mathfrak{k}_c$ and \mathfrak{h}_c , respectively. Then $X = G/H$ is called a *semisimple symmetric space* and $X^d = G^d/K^d$ is a Riemannian symmetric space of noncompact type. Both of X and X^d are "real forms" of a *complex symmetric space* $X_c = G_c/H_c$. (Remark. We don't have to assume that σ lifts to G_c in the following.)

Example 1. Let $G_c = SL(2, \mathbb{C})$, $G = SL(2, \mathbb{R})$, $\theta g = {}^t g^{-1}$ and

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_c.$$

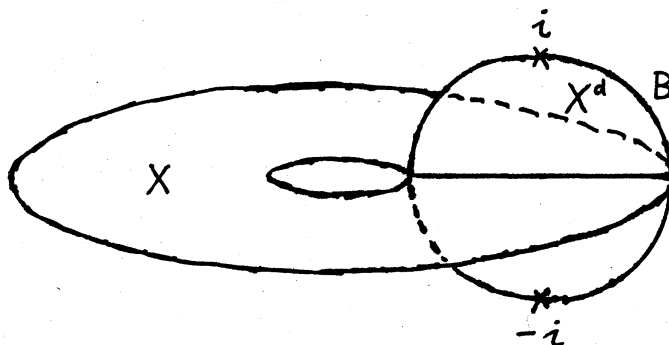
Then

$$K = SO(2), H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}_{>0} \right\},$$

$$K^d = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid |a| = 1 \right\}, H^d = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a \in \mathbb{R}_{>0}, b \in i\mathbb{R}, a^2 + b^2 = 1 \right\},$$

$$G^d = SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, a\bar{a} - b\bar{b} = 1 \right\},$$

X^d is identified with the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$ with the G^d -action



$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} z = \frac{az + b}{\bar{b}z + \bar{a}} \text{ for } \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G^d \text{ and } z \in X^d,$$

$X = G/H$ intersects with X^d on the real axis, and the H^d -orbits on the boundary $B = \{z \in \mathbb{C} \mid |z| = 1\}$ of X^d are $\{i\}, \{-i\}, \{z \in B \mid \operatorname{Re} z > 0\}$ and $\{z \in B \mid \operatorname{Re} z < 0\}$.

Let $\underline{A}_K(X)$ (resp. $\underline{A}_{H^d}(X^d)$) be the space of K -finite (resp. K_c -finite) analytic functions on X (resp. X^d). Here “a function f on X^d is K_c -finite” implies that it is H^d -finite and the representation of H^d on the space spanned by $H^d f$ lifts to a holomorphic representation of K_c . Then the analytic continuation in X_c gives an isomorphism $f \mapsto f^\eta$ of $\underline{A}_K(X)$ onto $\underline{A}_{H^d}(X^d)$ which commutes with the left \mathfrak{g}_c -action and the $\mathbf{D}(X)$ -action ([1]). Here $\mathbf{D}(X)$ is the ring of G -invariant differential operators on X . By the analytic continuation, $\mathbf{D}(X)$ is identified with $\mathbf{D}(X^d)$ the ring of G^d -invariant differential operators on X^d .

Let \mathfrak{a}^d be a maximal abelian subspace of $\mathfrak{s}^d = i(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{s} \cap \mathfrak{q})$ such that $\mathfrak{a} = \mathfrak{a}^d \cap \mathfrak{s}$ is maximal abelian in $\mathfrak{s} \cap \mathfrak{q}$. Let Σ^+ be a positive system of the root system $\Sigma(\mathfrak{a}^d)$ such that $\langle \Sigma^+, Y \rangle \subset \mathbb{R}_{\geq 0}$ for a generic Y in \mathfrak{a} . Put $A = \exp \mathfrak{a}$ and define a closed subset $A^+ = \{a \in A \mid a^\alpha \geq 1 \text{ for all } \alpha \in \Sigma^+\}$ of A . Let P be a minimal parabolic subgroup of G^d defined by

$$P = P(\mathfrak{a}^d, \Sigma^+) = M^d A^d N^d$$

where $M^d = Z_{K^d}(A^d)$ (the centralizer of A^d in K^d), $A^d = \exp \mathfrak{a}^d$, $\mathfrak{n}^d = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^d(\mathfrak{a}^d; \alpha)$ and $N^d = \exp \mathfrak{n}^d$. Put $J = N_{K^d}(A)/N_{K \cap H}(A)Z_{K^d}(A) = N_K(A)/N_{K \cap H}(A)Z_K(A)$ where $N_*(**)$ is the normalizer of $**$ in $*$.

Proposition 1 (c.f. [1]). $G = K(\bigcup_{m \in J} mA^+m^{-1})H$

Proposition 2 ([4]). $\{H^d mP \mid m \in J\}$ is the set of open H^d -orbits on G^d/P .

Let λ be a complex linear form on \mathfrak{a}_c^d and let $\underline{B}(G^d/P; L_\lambda)$ be the space of hyperfunctions on G^d satisfying $f(xman) = a^{\lambda-\rho} f(x)$ for $x \in G^d$, $m \in M^d$, $a \in A^d$ and $n \in N^d$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ ($m_\alpha = \dim \mathfrak{g}^d(\mathfrak{a}^d; \alpha)$). Suppose that $\operatorname{Re} \langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Sigma^+$. Then the Poisson transform p_λ defined by

$$(p_\lambda f)(x) = \int_{K^d} f(xk) dk = \int_{K^d} h(x^{-1}k)^{-\lambda-\rho} f(k) dk$$

(where $h : G^d \rightarrow A^d$ is the projection with respect to the Iwasawa decomposition $G^d = K^d A^d N^d$) gives a G^d -isomorphism of $\underline{B}(G^d/P; L_\lambda)$ onto $\underline{A}(X^d)_\lambda = \{f \in \underline{A}(X^d) \mid Df = \chi_\lambda(D)f \text{ for } D \in \mathbf{D}(X^d)\}$ ([2]). Here χ_λ is the character of $\mathbf{D}(X^d)$ parametrized by λ . Note that $\chi_\lambda = \chi_\nu \iff \nu \in W\lambda$ where $W = N_{K^d}(A^d)/Z_{K^d}(A^d)$ is the Weyl group of $\Sigma(\mathfrak{a}^d)$.

Let g be an H^d -finite element in $\underline{B}(G^d/P; L_\lambda)$. Then $V = \operatorname{supp} g$ is a closed H^d -invariant subset of G^d/P .

Example 2([1]). Let $V = H^d x_0 P = (K \cap H) x_0 P$ be a closed H^d -orbit on G^d/P . Define a distribution T on K^d/M^d by

$$\langle T, \varphi \rangle = \int_{K \cap H} \varphi(k x_0) dk \text{ for } \varphi \in C^\infty(K^d/M^d)$$

The distribution T is identified with an element T_λ of $B(G^d/P; L_\lambda)$ by the inclusion $K^d \hookrightarrow G^d$ and T_λ becomes H^d -finite under some condition on λ . Flensted-Jensen defined generating functions $\psi_\lambda \in \underline{A}_K(X)_\lambda = \{f \in \underline{A}_K(X) \mid Df = \chi_\lambda(D)f \text{ for } D \in \mathbf{D}(X)\}$ of discrete series for X by

$$\psi_\lambda^\eta(x) = (p_\lambda T_\lambda)(x) = \int_{K \cap H} h(x^{-1} k x_0) dk.$$

(Discrete series for X are the representations of G realized in subspaces of $L^2(X)_\lambda = \{f \in L^2(X) \mid Df = \chi_\lambda(D)f \text{ for } D \in \mathbf{D}(X)\}$ for some λ).

For V and $m \in J$, define a subset $W_{V,m} = \{w \in W \mid V(Pw^{-1}P)^d \supset H^d m P \text{ and } V(Pv^{-1}P)^d \not\supset H^d m P \text{ if } (Pv^{-1}P)^d \subsetneq (Pw^{-1}P)^d \text{ of } W\}$. Put $S_{V,m,\lambda} = W_{V,m}\lambda|_A$. Assume $\operatorname{Re}\langle \lambda, \alpha \rangle > 0$ ($\alpha \in \Sigma^+$) for simplicity in the following.

Theorem ([6]). Let $m \in J$, $f \in \underline{A}_K(X)_\lambda$ and put $V = V_f = \operatorname{supp} p_\lambda^{-1}(f^\eta)$. Then there exist nonzero analytic functions f_μ on K for all $\mu \in S_{V,m,\lambda}$ such that

$$f(kmam^{-1}H) = \sum_{\mu \in S_{V,m,\lambda}} f_\mu(k) a^{\mu-\rho} + o\left(\sum_{\mu \in S_{V,m,\lambda}} |a^{\mu-\rho}|\right)$$

($k \in K$) when $a^\alpha \rightarrow +\infty$ for all $\alpha \in \Sigma^+|_{\mathfrak{a}} \setminus \{0\}$.

Remark. Above formula gives the asymptotic behavior of f at the minimal boundaries of X . But we can see also the asymptotic behavior at other boundaries from this formula since we have expansions of f at these boundaries and the boundary values are analytic([6]).

Corollary ([6]). Let $f \in \underline{A}_K(X)_\lambda$. Then $f \in L^2(X) \iff$ (P) $|a^\mu| < 1$ for any $\mu \in \bigcup_{m \in J} S_{V_f,m,\lambda}$ and $a \in A^+ \setminus \{1\}$.

Lemma ([7] Lemma 7 + [3] Lemma 1.2). (P) \iff (i) $\operatorname{rank} X = \operatorname{rank}(K/K \cap H)$ and (ii) $V_f \subset$ the union of closed H^d -orbits on G^d/P . ((ii) $\iff \dim V_f$ is the smallest.)

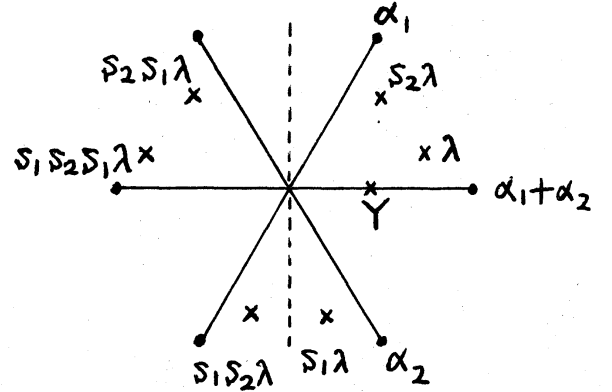
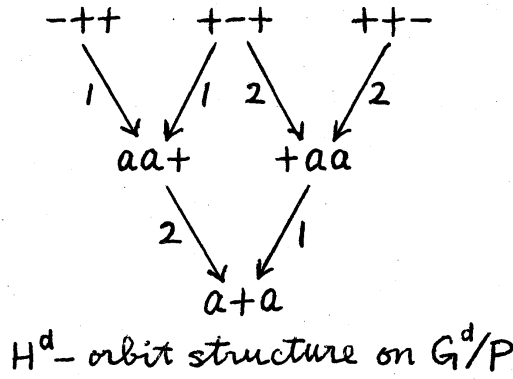
Remark. By the above corollary and the lemma, we don't need case-by-case checkings in [7] p.361-p.377.

Example 3. Let $X = G/H = SU(2, 1) \times SU(2, 1)/\text{diagonal} \cong SU(2, 1)$. Then $G^d \cong SL(3, \mathbb{C})$, $H^d \cong (GL(1, \mathbb{C}) \times GL(2, \mathbb{C})) \cap G^d$ and $\mathfrak{s}^d \cong \{\text{hermitian matrices in } \mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})\}$. Put

$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{a} = \mathbb{R}Y$ is a maximal abelian subspace of $\mathfrak{s}^d \cap \mathfrak{q}^d$ and $\mathfrak{a}^d = \mathfrak{z}_{\mathfrak{s}^d}(Y)$ is a maximal abelian subspace of \mathfrak{s}^d . Put $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} = \{\alpha \in \Sigma(\mathfrak{a}^d) \mid \alpha(Y) > 0\}$ and define a minimal parabolic subgroup P of G^d from \mathfrak{a}^d and Σ^+ . Let s_i denote the reflection with respect to α_i ($i = 1, 2$).

There are six H^d -orbits $V_1 = -++$, $V_2 = +-+$, $V_3 = ++-$, $V_4 = aa+$, $V_5 = +aa$ and $V_6 = H^d P = a+a$ on the flag manifold G^d/P where V_1, V_2 and V_3 are closed and V_6 is open ([5]). We can see that $W_{V_1,1} = \{s_2 s_1\}$, $W_{V_2,1} = \{s_2 s_1, s_1 s_2\}$ and $W_{V_3,1} = \{s_1 s_2\}$ from the diagram of the orbit structure. (The diagram implies that $V_1(P s_1 P)^d = V_1 \cup V_2 \cup V_4$, for instance.) We get easily that $\text{Re}(s_2 s_1 \lambda)(Y) < 0$ and $\text{Re}(s_1 s_2 \lambda)(Y) < 0$ from the assumption $\text{Re}\langle \lambda, \alpha_i \rangle > 0$ ($i = 1, 2$). Hence the property (P) holds for V_1, V_2 and V_3 . On the other hand, let V be a closed H^d -invariant subset of G^d/P such that $V \not\subset V_1 \cup V_2 \cup V_3$. Then $V \supset V_4$ or $V \supset V_5$ and therefore $W_{V,1} \ni s_2$ or $W_{V,1} \ni s_1$. Since $\text{Re}(s_i \lambda)(Y) > 0$, the property (P) does not hold. The discrete series coming from V_1 and V_3 are the holomorphic and anti-holomorphic discrete series for $X = SU(2, 1)$ and the one coming from V_2 is the other one.



References

- [1] M. Flensted-Jensen. Discrete series for semisimple symmetric spaces. *Ann. of Math.*, 111:253–311, 1980.
- [2] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka. Eigenfunctions of invariant differential operators on a symmetric space. *Ann. of Math.*, 107:1–39, 1978.
- [3] T. Matsuki. A description of discrete series for semisimple symmetric spaces II. *Advanced Studies in Pure Math.*, 14:531–540, 1988.
- [4] T. Matsuki. The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. *J. Math. Soc. Japan*, 31:331–357, 1979.

- [5] T Matsuki and T. Oshima. Embeddings of discrete series into principal series. In *The Orbit Method in Representation Theory*, pages 147–175, Birkhäuser, 1990.
- [6] T. Oshima. Asymptotic behavior of spherical functions on semisimple symmetric spaces. *Advanced Studies in Pure Math.*, 14:561–601, 1988.
- [7] T. Oshima and T. Matsuki. A description of discrete series for semisimple symmetric spaces. *Advanced Studies in Pure Math.*, 4:331–390, 1984.